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Generalization of D'Alembert's Method for Applications to Dynamics

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Introduction

A PROBLEM that sometimes arises in dynamics and control theory is the determination, as a function of time, of the position of a point P that moves on a closed, already known, approximately circular orbit. An example is a near-Earth spacecraft that moves in the gravitational, potential force field, perturbed by a nonpotential force such as the drag in the upper atmosphere. Because the combined force and the approximate orbit are assumed known, the force is known as a function of the angular coordinate (measured from an arbitrary, fixed radius from the origin O, the latter being chosen arbitrarily near the center of the orbit). Not yet known is the angular position θ of P as a function of time. Similar problems arise in near-circular limit cycles, hysteresis loops, or other problems that make use of the phase plane.

To relate the generalization in this Note to D'Alembert's solution of Kepler's equation, the latter is briefly recalled. In the notation usual in astronomy, Kepler's equation has the form $E - e \sin E = Nt$, where E is the eccentric anomaly, N the mean angular velocity, and e the orbit's eccentricity. The object is to find E as a function of time t and, from this, the position of the planet. In place of this transcendental equation, the same content can also be expressed by a nonlinear, first-order differential equation. D'Alembert's method of solution (see Ref. 1) consists in introducing a new parameter of smallness, $\varepsilon = E - Nt = e(\sin Nt \cos\varepsilon + \cos Nt \sin\varepsilon)$ and expanding ε in a power series in e, with coefficients that are functions of Nt.

Generalization of D'Alembert's Method

The generalization needed for more general problems relating to near-circular orbits consists in replacing the simple sine function in Kepler's equation by a Fourier series. The angle $\theta(t)$ is defined as before, and e is the parameter of smallness. The motions considered here are, therefore, limited to angular velocities about O that do not deviate too greatly from the constant angular velocity $\omega = 2\pi/T$, where T is the period. The difference between $\theta(t)$ and the uniformly increasing angle ωt is designated by $\gamma(t)$. As in the spacecraft problem, the lead angle γ has been calculated as a function of the angular position, but θ itself as a function of time is not yet known. Therefore,

$$\gamma(t) = eF[\theta(t)] = eF[\omega t + \gamma(t)] \tag{1}$$

where the function $F(\theta)$ is given. Equation (1) is a functional equation for $\gamma(t)$ and, therefore, also for the position of P on the orbit as a function of time.

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The function is periodic with period 2π and will be represented by a Fourier series.

Alternatively, Eq. (1) can also be written as a differential equation. If $f(\theta)$ is the derivative $dF/d\theta$, differentiating Eq. (1) by t, and observing that $d\gamma/dt = d\theta/dt - \omega$, one obtains

$$[1 - ef(\theta)]\frac{d\theta}{dt} = \omega \tag{2}$$

hence, a nonlinear differential equation for $\theta(t)$.

The function $f(\theta)$ in Eq. (2) is expressed by its Fourier series

$$f(\theta) = \sum_{n=1}^{\infty}[a_n \cos n\theta + b_n \sin n\theta], \qquad n = 1, 2, 3, \ldots \tag{3a}$$

The coefficients are known because $F(\theta)$ is given. Integrating term by term,

$$F(\theta) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{n}\right)a_n \sin n\theta - \left(\frac{1}{n}\right)b_n \cos n\theta\right] \tag{3b}$$

Without loss of generality, the constant terms in Eqs. (3a) and (3b) can be dispensed with because the mean of γ over the orbit is zero. Therefore,

$$\gamma(t) = e\sum_{n=1}^{\infty}\left\{\left(\frac{1}{n}\right)a_n \sin n[\omega t + \gamma(t)] - \left(\frac{1}{n}\right)b_n \cos n[\omega t + \gamma(t)]\right\} \tag{4}$$

By Eq. (1), $\gamma(t)$ here is the as yet unknown perturbation term. Expressed in the form of a power series in e,

$$\gamma(t) = \alpha_1(t)e + \alpha_2(t)e^2 + \alpha_3(t)e^3 + \cdots \tag{5}$$

It remains to determine the factors $\alpha_i(t)$, $i = 1, 2, 3, \ldots$, from which in turn $\gamma(t)$ can be found.

From the trigonometric identity for sums of angles,

$$\begin{aligned}\sin n[\omega t + \gamma(t)] &= \sin n\omega t \cos n\left(\alpha_1 e + \alpha_2 e^2 + \cdots\right) \\ &\quad + \cos n\omega t \sin n\left(\alpha_1 e + \alpha_2 e^2 + \cdots\right) \\ &= (\sin n\omega t)\left[1 - \frac{n^2\left(\alpha_1 e + \alpha_2 e^2 + \cdots\right)^2}{2!} + \cdots\right] \\ &\quad + (\cos n\omega t)\left[\frac{n\left(\alpha_1 e + \alpha_2 e^2 + \cdots\right)}{1!} - \cdots\right]\end{aligned}$$

and analogously for $\cos n[\omega t + \gamma(t)]$. From Eqs. (4) and (5),

$$\alpha_1 + \alpha_2 e + \alpha_3 e^2 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n} \left\{ (\sin n\omega t) \left[1 - \frac{n^2 (\alpha_1 e + \alpha_2 e^2 + \cdots)^2}{2!} + \cdots \right] + (\cos n\omega t) \left[\frac{n(\alpha_1 e + \alpha_2 e^2 + \cdots)}{1!} - \cdots \right] \right\}$$

$$- \sum_{n=1}^{\infty} \frac{b_n}{n} \left\{ (\cos n\omega t) \left[1 - \frac{n^2 (\alpha_1 e + \alpha_2 e^2 + \cdots)^2}{2!} + \cdots \right] - (\sin n\omega t) \left[\frac{n(\alpha_1 e + \alpha_2 e^2 + \cdots)}{1!} - \cdots \right] \right\}$$

Next, all terms of equal power of e are collected, with the result that

$$\alpha_1 = \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin n\omega t - \frac{b_n}{n} \cos n\omega t \right)$$

$$\alpha_2 = \alpha_1 \sum_{n=1}^{\infty} \left(\frac{a_n}{1!} \cos n\omega t + \frac{b_n}{1!} \sin n\omega t \right)$$

$$\alpha_3 = \alpha_1^2 \sum_{n=1}^{\infty} \left(-\frac{n a_n}{2!} \sin n\omega t + \frac{n b_n}{2!} \cos n\omega t \right) + \alpha_2 \sum_{n=1}^{\infty} \left(\frac{a_n}{1!} \cos n\omega t + \frac{b_n}{1!} \sin n\omega t \right)$$

$$\cdots \tag{6}$$

As is evident from these equations, in general, the coefficients can be calculated recursively from each other. (We reproduce here only the expressions for α_1, α_2, and α_3. More extensive results, directly for $\gamma(t)$, up to order 4, are given in Table 1).

As the final result, a series is given for calculating directly the angle $\gamma(t)$. The derivation, even of just the first four terms in the general expansion in terms of the parameter of smallness e is lengthy. For this reason, we reproduce here only the final result (Table 1).

The convergence properties of the series are not known. Only in the special case of D'Alembert's original problem is it known that the series converges for all frequencies, provided that $e < 0.6627$, a result due to Plummer.[2] A numerical indication of the rate of convergence is stated in Example 2 to follow.

Calculating from Eq. (6) the coefficients α_i and substituting the results into Eq. (5), one finds

$$\begin{aligned} \gamma(t) = {} & (e P_{11} + e^2 P_{12} + e^3 P_{13} + \cdots) \sin \omega t \\ & + (e Q_{11} + e^2 Q_{12} + e^3 Q_{13} + \cdots) \cos \omega t \\ & + (e P_{21} + e^2 P_{22} + e^3 P_{23} + \cdots) \sin 2\omega t \\ & + (e Q_{21} + e^2 Q_{22} + e^3 Q_{23} + \cdots) \cos 2\omega t + \cdots \end{aligned} \tag{7}$$

where the multinomials P_{ij} and Q_{ij} are as given in Table 1. Table 1 suffices for including in the Fourier series terms up to $m = 4$, where m is the sum of the indices of the Fourier coefficient products (for instance, $m = 3$ for $a_1^2 b_1$ or for $a_1 b_2$, or $m = 4$ for $a_2 b_1^2$, etc.).

Table 1 Coefficients P_{ij} and Q_{ij} up to orders e^4 and $m = 4$

Coefficient P_{ij}	Coefficient Q_{ij}
$P_{11} = a_1$	$Q_{11} = -b_1$
$P_{12} = -\left(\frac{1}{4}\right)(a_1 a_2 + b_1 b_2)$	$Q_{12} = \left(\frac{1}{4}\right)(a_1 b_2 - a_2 b_1)$
$P_{13} = -\left(\frac{1}{8}\right) a_1 \left(a_1^2 + b_1^2\right)$	$Q_{13} = \left(\frac{1}{8}\right) b_1 \left(a_1^2 + b_1^2\right)$
$P_{14} = 0$	$Q_{14} = 0$
$P_{21} = \left(\frac{1}{2}\right) a_2$	$Q_{21} = -\left(\frac{1}{2}\right) b_2$
$P_{22} = \left(\frac{1}{2}\right)\left(a_1^2 - b_1^2\right) - \left(\frac{1}{3}\right)(a_1 a_3 + b_1 b_3)$	$Q_{22} = -a_1 b_1 + \left(\frac{1}{3}\right)(a_1 b_3 - a_3 b_1)$
$P_{23} = -\left(\frac{1}{2}\right) a_2 \left(a_1^2 + b_1^2\right)$	$Q_{23} = \left(\frac{1}{2}\right) b_2 \left(a_1^2 + b_1^2\right)$
$P_{24} = -\left(\frac{1}{6}\right)\left(a_1^4 - b_1^4\right)$	$Q_{24} = \left(\frac{1}{3}\right) a_1 b_1 \left(a_1^2 + b_1^2\right)$
$P_{31} = \left(\frac{1}{3}\right) a_3$	$Q_{31} = -\left(\frac{1}{3}\right) b_3$
$P_{32} = \left(\frac{3}{4}\right)(a_1 a_2 - b_1 b_2)$	$Q_{32} = -\left(\frac{3}{4}\right)(a_1 b_2 + a_2 b_1)$
$P_{33} = \left(\frac{3}{8}\right) a_1 \left(a_1^2 - 3b_1^2\right)$	$Q_{33} = \left(\frac{3}{8}\right) b_1 \left(b_1^2 - 3a_1^2\right)$
$P_{34} = 0$	$Q_{34} = 0$
$P_{41} = \left(\frac{1}{4}\right) a_4$	$Q_{41} = -\left(\frac{1}{4}\right) b_4$
$P_{42} = \left(\frac{1}{4}\right)\left(a_2^2 - b_2^2\right) + \left(\frac{2}{3}\right)(a_1 a_3 - b_1 b_3)$	$Q_{42} = -\left(\frac{1}{2}\right) a_2 b_2 - \left(\frac{2}{3}\right)(a_1 b_3 + a_3 b_1)$
$P_{43} = a_2 \left(a_1^2 - b_1^2\right) - 2 a_1 b_1 b_2$	$Q_{43} = -b_2 \left(a_1^2 - b_1^2\right) - 2 a_1 a_2 b_1$
$P_{44} = \left(\frac{1}{3}\right)\left(a_1^4 + b_1^4\right) - 2 a_1^2 b_1^2$	$Q_{44} = -\left(\frac{4}{3}\right) a_1 b_1 \left(a_1^2 - b_1^2\right)$

Examples

1) Kepler's equation in its usual form is obtained by replacing in Eq. (1) the function F by the sine function and $\omega t + \gamma$ by E. Hence, in Eq. (4) all Fourier coefficients vanish, except that $a_1 = 1$. Therefore, making use of Table 1, one obtains the well-known result

$$\gamma(t) = \left[e - \left(\tfrac{1}{8}\right) e^3 \right] \sin \omega t + \left[\left(\tfrac{1}{2}\right) e^2 - \left(\tfrac{1}{6}\right) e^4 \right] \sin 2\omega t + \left(\tfrac{3}{8}\right) e^3 \sin 3\omega t + \left(\tfrac{1}{3}\right) e^4 \sin 4\omega t + \cdots \tag{8}$$

2) Suppose that the angle $\gamma(t)$ leads the uniformly increasing angle ωt by $\gamma(t) = e F[\theta(t)]$, where F is a known periodic function of its argument θ, but where θ as a function of time is not yet known. For instance, if

$$F(\theta) = \frac{4}{\pi} \left(\sin \theta - \frac{\sin 3\theta}{3^2} + \frac{\sin 5\theta}{5^2} - \cdots \right)$$

$F(\theta)$ represents a periodic sequence of triangular ramps.

In this case, only a few of the multinomials in Table 1 differ from zero. Equation (7) now becomes

$$\begin{aligned} \gamma(t) = {} & \left[(4/\pi) e - \left(\tfrac{1}{8}\right)(4/\pi)^3 e^3 + \cdots \right] \sin \omega t \\ & + \left[\left(\tfrac{11}{18}\right)(4/\pi)^2 e^2 - \cdots \right] \sin 2\omega t \\ & + \left[-\left(\tfrac{1}{9}\right)(4/\pi) e + \left(\tfrac{3}{8}\right)(4/\pi)^3 e^3 - \cdots \right] \sin 3\omega t \\ & + \left[-\left(\tfrac{2}{9}\right)(4/\pi)^2 e^2 + \cdots \right] \sin 4\omega t + \cdots \end{aligned} \tag{9}$$

For instance, if $e = 0.1$, the coefficient multiplying $\sin(\omega t)$ to lowest order is 0.1273; to the next order it is 0.1270. If $e = 0.2$, the corresponding values are 0.2546 and 0.2525. If $e = 0.3$, they are 0.3820 and 0.3750.

Conclusions

D'Alembert's method for solving Kepler's equation has been generalized for other applications to dynamics that require the solution of a functional equation. Numerical applications are made simple by means of the coefficients listed in Table 1.

References

[1]Cornelisse, J. W., Schoeyer, H. F. R., and Wakker, K. F., *Rocket Propulsion and Spaceflight Dynamics*, Pitman, London, 1979, p. 376.

[2]Plummer, H. C., *An Introductory Treatise on Dynamical Astronomy*, new ed., Dover, New York, 1990, p. 137.